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Full asymptotic expansion of transition probabilities in the adiabatic limit*

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Abstract. We consider a two-level quantum mechanical system driven by an analytic time-dependent Hamiltonian of the form $H(\varepsilon t)$. In the adiabatic limit, $\varepsilon \ll 1$, the transition probability $\mathscr{P}(+,-)$ from one energy level (labelled by $-$) at time $t=-\infty$ to the other (labelled by $+$) at time $t=+\infty$ is known to behave as $\mathscr{P}(+,-)=\exp(-\alpha_{-1}\varepsilon^{-1})\exp(\alpha_0)(1+\mathrm{O}(\varepsilon))$. Using a simple iterative procedure generating Hamiltonians $H_0=H, H_1, \ldots, H_{N+1}$, we compute the full asymptotic expansion of the transition probability

$$\ln \mathscr{P}(+,-) = -\alpha_{-1}\varepsilon^{-1} + \alpha_0 + \sum_{j=0}^{N} \alpha_j \varepsilon^j + \mathrm{O}(\varepsilon^{N+1}) \qquad \forall N \geqslant 0.$$

1. Introduction

The adiabatic theorem of quantum mechanics has a long history since it had already been established by Born and Fock in 1928 [1]. This theorem describes the asymptotic regime of the slow evolution of a quantum mechanical system. Consider the Schrödinger equation (with $\hbar = 1$)

$$\mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}\tau}(\tau) = H(\varepsilon\tau)\psi(\tau). \tag{1.1}$$

The parameter $1/\varepsilon$ is the characteristic time-scale of the system. In terms of the rescaled time $t=\varepsilon\tau$ the equation reads

$$\mathrm{i}\varepsilon\frac{\mathrm{d}\psi_\varepsilon}{\mathrm{d}t}(t) = H(t)\psi_\varepsilon(t). \tag{1.2}$$

The adiabatic limit corresponds to the singular limit $\varepsilon \to 0$ which describes an infinitely slow evolution. In this limit, if $H(t)$ has an energy level $e(t)$ isolated for all $t \in [t_0, t_1]$, then a system in the eigenstate corresponding to $e(t_0)$ at t_0 evolves to an eigenstate corresponding to $e(t_1)$ at t_1. In particular, a transition to an eigenstate with an energy different from $e(t_1)$ is forbidden.

The natural task which comes next is to estimate this transition probability when ε is small but finite. It has been shown under a very general hypothesis that the transition probability tends to zero as ε^2 [2]. This gives a bound to the leading term of the asymptotic behaviour.

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When the Hamiltonian is C^∞-smooth and $\mathrm{d}^n/\mathrm{d}t^n\, H(t)|_{t=t_0}=0$ for all n one can write an asymptotic expansion in powers of ε for the transition probability at time t_1. This result was first obtained by Lenard [3] for matrices and has been generalized in a different way by Garrido [4], Nenciu [5] and Joye and Pfister [6] using the iterative scheme of section 2. An important feature of this asymptotic expansion is that the first N terms ($N\geqslant 2$) vanish if $\mathrm{d}^n/\mathrm{d}t^n\, H(t)|_{t=t_1}=0$ for all $n\leqslant N$, showing that in such a case the transition probability at time t_1 is of order $\mathrm{O}(\varepsilon^{N+1})$. An interesting case occurs if $\mathrm{d}^n/\mathrm{d}t^n\, H(t)|_{t=t_0}=\mathrm{d}^n/\mathrm{d}t^n\, H(t)|_{t=t_1}=0$ for all n, since the transition probability at time t_1 is smaller than $\mathrm{O}(\varepsilon^N)$ for all N.

This result can be improved if the Hamiltonian $H(t)$ depends analytically on time in a region including the real axis. The analyticity assumption and the above conditions on the derivatives of $H(t)$ at t_0 and t_1 are compatible if we choose $t_0=-\infty$ and $t_1=+\infty$. It can be shown that they are realized provided $H(t)$ is analytic in a strip including the real axis and $H(t)$ has well-defined limits $H(+)$ and $H(-)$ when $t\to+\infty$ and $t\to-\infty$. Under the weak hypothesis that the limit of $|t|^{1+\alpha}(H(t)-H(\pm))$ is zero when $t\to\pm\infty$ for some $\alpha>0$, it has been shown by Joye and Pfister [7] that the transition probability at time $t_1=+\infty$ is bounded by $\mathrm{e}^{-\delta/\varepsilon}$, $\delta>0$. Similar results have been derived recently in a different way by Nenciu [8]. All these results hold for very general Hamiltonians. We should also mention the paper by Jaksic and Segert [9] on the same subject; however, their results are weaker. This behaviour was expected to be true for a long time. In the case of an analytic 2×2 matrix, real symmetric on the real axis, Dykhne [10] proposed in 1962 a formula for the exponentially small leading term of the transition probability at $t_1=+\infty$. A proof of this formula was given in 1977 by Hwang and Pechukas [11]. In the general case of an analytic 2×2 matrix, Hermitian on the real axis, the Dykhne formula must be completed by a prefactor of geometrical nature as observed independently by Berry [12] and Joye *et al* [13]. A detailed analysis as well as a geometrical interpretation of the Dykhne formula is given in [13]. Finally we would like to mention two recent works by Hagedorn on related topics [14] and [15]. In [14] the author gives an asymptotic expansion in ε of the wavefunction in the presence of a *real* eigenvalue-crossing point. In the second paper the transition probability is computed in the limit $\varepsilon\to 0$ for a system having an avoided crossing with a gap of order $\sqrt{\varepsilon}$. In this scaling limit the end result is given by the Landau-Zener formula. Notice that there is no geometric factor in this case.

In this paper we derive a generalization of the Dykhne formula which allows us to write an asymptotic expansion in powers of ε at any order for the logarithm of the transition probability at $t_1=+\infty$. The main point of the proof is to combine the iterative scheme of [6] in order to get corrections up to order ε^N and to use the method of [13] to estimate the higher-order terms. The idea to apply the iterative scheme of Garrido to derive an asymptotic expansion of the Berry phase was already used by Berry [16]. This procedure has been further emphasized in a recent work by Berry [17] where it is called superadiabatic renormalization. In these two references the nature of the asymptotic expansion is studied and the results should apply to our expansion (1.12). With the intent of stating our results in a precise way, let us describe what kinds of two-level system are considered. They can be characterized by four conditions.

(I) Analyticity. There exists a closed strip $S_a=\{z=t+\mathrm{i}s\in\mathbb{C}\,|\,|s|\leqslant a\}$ such that the Hamiltonian $H(t)$, $t\in\mathbb{R}$ is a 2×2 Hermitian traceless matrix which has an analytic extension on some domain containing S_a.

Remark. The important properties of S_a which we use are that it is a connected, simply connected set containing $\mathbb{R}$ and coincides with a strip when $|t|$ is large enough. It is convenient to suppose that S_a is closed.

(II) Behaviour at infinity. There exist 2×2 non-degenerate Hermitian matrices $H(+)$ and $H(-)$ such that

$$\lim_{t\to\pm\infty}\sup_{|s|<a}\|H(t+\mathrm{i}s)-H(\pm)\|\,|t|^{1+\alpha}=0$$

for some positive α.

(III) Separation of the spectrum. For each $t\in\mathbb{R}$ the spectrum of $H(t)$ consists of two separated eigenvalues $e^+(t)$ and $e^-(t)$ such that $e^+(t)-e^-(t)\geq\delta$, $\delta>0$.

The condition on the trace can always be satisfied by an energy shift. A convenient way of looking at such systems is to write them as spin systems in a time-dependent magnetic field

$$\begin{aligned} H(z) &= \boldsymbol{B}(z)\cdot \boldsymbol{s} \\ &\equiv B_1(z)\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+B_2(z)\frac{1}{2}\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}+B_3(z)\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{aligned} \tag{1.3}$$

the functions B_k being analytic in S_a, $B_k(\bar z)=\overline{B_k(z)}$ and for some positive α

$$\lim_{t\to\pm\infty}\sup_{|s|\leq a}|B_k(t+\mathrm{i}s)-B_k(\pm)|\,|t|^{1+\alpha}=0. \tag{1.4}$$

By the Cauchy formula we have for any $a'<a$ and integer $n\geq 1$

$$\lim_{t\to\pm\infty}\sup_{|s|\leq a'}\left|\frac{\mathrm{d}^n}{\mathrm{d}z^n}B_k(t+\mathrm{i}s)\right||t|^{1+\alpha}=0. \tag{1.5}$$

Remark. By choosing the constant a slightly smaller we may suppose that (1.5) is true with $a'\leq a$.

The eigenvalues on the real axis are

$$e^{\pm}(t)=\pm\tfrac{1}{2}\sqrt{\rho(t)} \tag{1.6}$$

where

$$\rho(t)=B_1^2(t)+B_2^2(t)+B_3^2(t) \tag{1.7}$$

is strictly positive. By convention we choose in (1.6) the branch of the square root which is positive on the positive real axis. The corresponding eigenprojections are

$$P^{\pm}(t)=\tfrac{1}{2}\mathbb{1}\pm\frac{\boldsymbol{B}(t)\cdot\boldsymbol{s}}{\sqrt{\rho(t)}}. \tag{1.8}$$

The eigenvalues and eigenprojections on S_a are defined by the analytic continuations of (1.6) and (1.8). They are multivalued and singular at the eigenvalue crossings which coincide with the zeros of the analytic continuation $\rho(z)$ of $\rho(t)$ in S_a. Notice that $\rho(z)$ is single valued in S_a. We suppose furthermore:

(IV) Eigenvalue crossings. The set X of zeros of $\rho(z)$ in S_a consists of $2n$ interior points $z_1, \bar z_1, \ldots, z_n, \bar z_n$ and each zero is simple. By convention $\operatorname{Im} z_k>0$, $k=1,\ldots,n$.

Let ψ_ε be the solution of the Schrödinger equation (1.2) satisfying the boundary condition

$$\lim_{t\to-\infty} \|P^-(t)\psi_\varepsilon(t)\| = 1 \tag{1.9}$$

which is equivalent, since ψ_ε is normalized, to the condition

$$\lim_{t\to-\infty} \|P^+(t)\psi_\varepsilon(t)\| = 0. \tag{1.10}$$

The problem which is solved in this paper is the following: to find an asymptotic expansion in powers of ε for the logarithm of the transition probability

$$\mathscr{P}(+,-) = \lim_{t\to+\infty} \|P^+(t)\psi_\varepsilon(t)\|^2 \tag{1.11}$$

at time $t=+\infty$. We show in section 2 that for any integer $N\geqslant 0$ there exists $\varepsilon^*(N)$ such that

$$\ln \mathscr{P}(+,-) = -\alpha_{-1}\varepsilon^{-1} + \sum_{k=0}^{N} \alpha_k\varepsilon^k + \mathrm{O}(\varepsilon^{N+1}) \tag{1.12}$$

provided $0\leqslant\varepsilon\leqslant\varepsilon^*(N)$. The term α_{-1} is positive and both terms α_{-1} and α_0 can be interpreted geometrically. They are given explicitly in (2.35) and (2.37). When we put $N=0$ in (1.12) we recover the generalized Dykhne formula found by Berry [12] and Joye *et al* [13]. The result (1.12) is proved in all cases for which we can prove the generalized Dykhne formula *as* in [13]. This means that another additional condition (condition V) is needed. This condition is analysed thoroughly in [13] and is recalled below in its geometrical version for the sake of completeness.

For each Hamiltonian $H(z)=\boldsymbol{B}(z)\cdot\boldsymbol{s}$ on S_a we can associate a distance d_ρ on S_a, where $\rho(z)=B_1^2(z)+B_2^2(z)+B_3^2(z)$. Let γ be some rectifiable curve in S_a. The *ρ-length* of γ is

$$|\gamma|_\rho = \int_\gamma |\rho(z)|^{1/2}|\mathrm{d}z| \tag{1.13}$$

and the *ρ-distance* $\mathrm{d}_\rho(z_1,z_2)$ between two points z_1 and z_2 of S_a is given by the infimum of $|\gamma|_\rho$ where γ is a rectifiable curve in S_a from z_1 to z_2. Having a distance d_ρ, we can introduce the concept of a *ρ-geodesic* which is a curve $t\mapsto\gamma(t)$ which is locally shortest for the ρ-length: if $z_1=\gamma(t_1)$ and $z_2=\gamma(t_2)$ and $|t_1-t_2|$ is small enough, then the curve between z_1 and z_2 which has the minimal ρ-length is given by $\gamma(t)$, $t\in[t_1,t_2]$. The supplementary condition is:

(V) Existence of an infinite geodesic. There exist an eigenvalue crossing, say z_1, and a geodesic $t\mapsto\gamma(t)$, $t\in\mathbb{R}$ in S_a, passing through z_1 such that $\lim_{t\to\pm\infty}\operatorname{Re}\gamma(t)=\pm\infty$ and $|\operatorname{Im}\gamma(t)|<a$ for large enough $|t|$.

In the Dykhne formula, the exponential rate of the transition probability is computed using one particular complex eigenvalue crossing (see (2.35)). In [13] we show that whenever condition V holds, the relevant eigenvalue crossing which is used in the Dykhne formula is the z_1 of condition V. We emphasize that z_1 is not necessarily the closest eigenvalue crossing to the real axis in the Euclidean distance, it is the closest eigenvalue crossing to the real axis in the ρ-distance. There is no known proof of the Dykhne formula without condition V, which is a condition of *global* character. There is another formulation of it in terms of Stokes lines and anybody who is familiar with WKB analysis will recognize that our condition is typical in such a context. We refer to [13] for more details and examples. We finish this section with one remark. The introduction of the ρ-distance and the ρ-geometry is one new important result of [13]. It allows, in particular, the eigenvalue crossing which governs the transition probability $\mathcal{P}(+,-)$ to be distinguished.

2. Asymptotic expansion of $\mathcal{P}(+,-)$

Let us now outline the strategy which is used for the proof of (1.12) and show how one computes the coefficients α_k. The technical details are treated in the next section. We assume that z_1 is the closest zero of $\rho(z)$ to the real axis in the ρ-distance and that condition V is fulfilled. (If we want to analyse the transition probability $\mathcal{P}(-,+)$ where the roles of the projections P^+ and P^- are exchanged, then $\overline{z_1}$ is the relevant eigenvalue crossing.) The main result of this section is formula (2.34).

We first introduce a sequence of Hamiltonians $H_0=H, H_1, \ldots$ as in the papers of Garrido [4] and Nenciu [5]; however, our construction is different and simpler.

Let

$$K_0(z)=\mathrm{i}(P^{+\prime}(z)P^+(z)+P^{-\prime}(z)P^-(z))=-\frac{\boldsymbol{B}(z)\wedge\boldsymbol{B}'(z)}{\rho(z)}\cdot\boldsymbol{s} \tag{2.1}$$

where $'$ denotes here and henceforth in this paper $\mathrm{d}/\mathrm{d}z$ and $\boldsymbol{B}\wedge\boldsymbol{B}'$ is the vector product of $\boldsymbol{B}$ and $\boldsymbol{B}'$. We define

$$H_1(z,\varepsilon)=H(z)-\varepsilon K_0(z)\equiv\boldsymbol{B}_1(z,\varepsilon)\cdot s. \tag{2.2}$$

The 2×2 matrix $H_1(z,\varepsilon)$ is meromorphic in S_a, all its poles, if any, are of first order and at points of X. It is still traceless since for any projection $Q(z)$ we have

$$Q(z)Q'(z)Q(z)=0 \tag{2.3}$$

and it satisfies condition II with the same $H(\pm)$, as well as condition III with $\delta/2$ instead of δ, provided ε is small enough (see lemma 2.2 below). Let r be some fixed small positive number (whose definite value is fixed in the proof of lemma 2.3), and let $D(z_0,r)=\{z\,|\,|z_0-z|<r\}$ be the disc of centre z_0 and radius r. We define Ω by the closed set

$$\Omega=S_a\setminus\bigcup_{z_0\in X}D(z_0,r) \tag{2.4}$$

and choose r so small that for any two points z_k and z_l of X the discs $\overline{D(z_k,r)}$ and $\overline{D(z_l,r)}$ are disjoint, each disc $\overline{D(z_k,r)}\subset S_a$ and does not intersect the real axis.

The eigenvalues of $H_1(z,\varepsilon)$ at $z\in\Omega$ are $\pm\frac{1}{2}\sqrt{\rho_1(z,\varepsilon)}$ where

$$\rho_1(z,\varepsilon)=B_{1,1}^2(z,\varepsilon)+B_{1,2}^2(z,\varepsilon)+B_{1,3}^2(z,\varepsilon) \tag{2.5}$$

is analytic and single valued in Ω. We define $e_1^{\pm}(t,\varepsilon)=\pm\frac{1}{2}\sqrt{\rho_1(t,\varepsilon)}$ for all $t\in\mathbb{R}$ and then $e_1^{\pm}(z,\varepsilon)$ by analytic continuation. The corresponding eigenprojections are $P_1^{\pm}(z,\varepsilon)$.

Lemma 2.1. If ε is small enough, then $H_1(z,\varepsilon)$ is analytic on Ω. There are no eigenvalue crossings of H_1 in Ω. The variation (in the positive sense) of the argument of $\rho_1(z,\varepsilon)$ around the boundary of any disc $D(z_0,r)$, $z_0\in X$, is 2π.

The last property mentioned in lemma 2.1 implies that under analytic continuation around a simple loop in Ω encircling exactly one disc $D(z_0, r)$, $z_0 \in X$, the eigenvalues of $H_1(z, \varepsilon)$ are exchanged. Let

$$K_1(z, \varepsilon) = \mathrm{i}(P_1^{+\prime}(z, \varepsilon)P_1^+(z, \varepsilon) + P_1^{-\prime}(z, \varepsilon)P_1^-(z, \varepsilon)). \tag{2.6}$$

We define

$$H_2(z, \varepsilon) = H(z) - \varepsilon K_1(z, \varepsilon). \tag{2.7}$$

By recurrence we introduce as above

$$\begin{aligned} K_{q-1}(z, \varepsilon) &= \mathrm{i}(P_{q-1}^{+\prime}(z, \varepsilon)P_{q-1}^+(z, \varepsilon) + P_{q-1}^{-\prime}(z, \varepsilon)P_{q-1}^-(z, \varepsilon)) \\ &= -\frac{\boldsymbol{B}_{q-1}(z, \varepsilon) \wedge \boldsymbol{B}_{q-1}'(z, \varepsilon)}{\rho_{q-1}(z, \varepsilon)} \cdot \boldsymbol{s} \end{aligned} \tag{2.8}$$

and

$$H_q(z, \varepsilon) = H(z) - \varepsilon K_{q-1}(z, \varepsilon). \tag{2.9}$$

Lemma 2.2. For any $q \geq 0$ there exists a positive $\varepsilon_*(q)$ and an integrable function $\beta_q(t)$ such that for any $z = t + \mathrm{i}s \in \Omega$ and any ε, $0 \leq \varepsilon \leq \varepsilon_*(q)$

$$\|K_q(t + \mathrm{i}s, \varepsilon)\| \leq \beta_q(t)$$

and

$$\|K_q(t + \mathrm{i}s, \varepsilon) - K_{q-1}(t + \mathrm{i}s, \varepsilon)\| \leq \varepsilon^q \beta_q(t)$$

where $K_{-1} = 0$ by convention.

Moreover lemma 2.1 is valid for H_q if $0 \leq \varepsilon \leq \varepsilon_*(q)$.

We want to prove (1.12). For this purpose we work with H_q, $q = N+1$, and we decompose $\psi_\varepsilon(t)$ in a basis of analytic eigenvectors of $H_q(t, \varepsilon)$, $\varphi_q^+(t, \varepsilon)$ and $\varphi_q^-(t, \varepsilon)$. We choose the phase of $\varphi_q^\sigma(t, \varepsilon)$ so that $\lim_{t \to -\infty} \varphi_q^\sigma(t, \varepsilon) = \varphi^\sigma(-\infty)$, where $\varphi^\sigma(-\infty)$ are fixed eigenvectors of $H(-\infty)$, and

$$\left\langle \varphi_q^\sigma(t, \varepsilon) \middle| \frac{\mathrm{d}}{\mathrm{d}t} \varphi_q^\sigma(t, \varepsilon) \right\rangle = 0 \qquad \sigma = \pm. \tag{2.10}$$

Here $\langle | \rangle$ is the usual Hermitian scalar product of $\mathbb{C}^2$. Condition (2.10) is equivalent to (see e.g. [4])

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_q^\sigma(t, \varepsilon) = -\mathrm{i} K_q(t, \varepsilon) \varphi_q^\sigma(t, \varepsilon) \qquad \sigma = \pm. \tag{2.11}$$

Let

$$\lambda_q^\sigma(t, \varepsilon) = \int_0^t e_q^\sigma(t', \varepsilon)\, \mathrm{d}t' \qquad \sigma = \pm \tag{2.12}$$

and let $\psi_\varepsilon(t)$ be decomposed as

$$\psi_\varepsilon(t) = \sum_{\sigma = \pm} c_q^\sigma(t, \varepsilon) \exp(-\mathrm{i}\varepsilon^{-1} \lambda_q^\sigma(t, \varepsilon)) \varphi_q^\sigma(t, \varepsilon). \tag{2.13}$$

We rewrite the differential equation

$$\mathrm{i}\varepsilon \frac{\mathrm{d}\psi_\varepsilon}{\mathrm{d}t}(t) = (H_q(t, \varepsilon) + \varepsilon K_{q-1}(t, \varepsilon)) \psi_\varepsilon(t) \tag{2.14}$$

as

$$\sum_{\sigma=\pm}\left(\frac{\mathrm{d}}{\mathrm{d}t}c_q^\sigma \exp(-\mathrm{i}\varepsilon^{-1}\lambda_q^\sigma)\varphi_q^\sigma + c_q^\sigma \exp(-\mathrm{i}\varepsilon^{-1}\lambda_q^\sigma)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_q^\sigma + \mathrm{i}c_q^\sigma \exp(-\mathrm{i}\varepsilon^{-1}\lambda_q^\sigma)K_{q-1}\varphi_q^\sigma\right)=0. \tag{2.15}$$

Taking the scalar product of this expression with φ_q^σ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}c_q^\sigma(t,\varepsilon)=\sum_{\tau=\pm} a_q^{\sigma,\tau}(t,\varepsilon)\exp[\mathrm{i}\varepsilon^{-1}(\lambda_q^\sigma(t,\varepsilon)-\lambda_q^\tau(t,\varepsilon))]c_q^\tau(t,\varepsilon) \tag{2.16}$$

with

$$a_q^{\sigma,\tau}(t,\varepsilon)=\mathrm{i}\langle\varphi_q^\sigma(t,\varepsilon)|(K_q(t,\varepsilon)-K_{q-1}(t,\varepsilon))\varphi_q^\tau(t,\varepsilon)\rangle. \tag{2.17}$$

By lemma 2.2

$$|a_q^{\sigma,\tau}(t,\varepsilon)|\leqslant \varepsilon^q\beta_q(t) \tag{2.18}$$

and this implies in particular that the coefficients $c_q^\sigma(t,\varepsilon)$ have well-defined limits $c_q^\sigma(\pm\infty,\varepsilon)$ when $t\to\pm\infty$, although $\psi_\varepsilon(t)$ does not have limits. The boundary condition (1.9) which is satisfied by ψ_ε is equivalent to

$$c_q^-(-\infty,\varepsilon)=1 \qquad \text{and} \qquad c_q^+(-\infty,\varepsilon)=0 \tag{2.19}$$

and the transition probability $\mathcal{P}(+,-)$ is given by

$$\mathcal{P}(+,-)=|c_q^+(+\infty,\varepsilon)|^2 \tag{2.20}$$

since $H_q(t)$ tends to $H(+)$ when $t\to\infty$ by lemma 2.2.

We now use the analyticity assumption in an essential way. On the simply connected domain S_a the solution $\psi_\varepsilon(t)$ has a single-valued analytic extension $\psi_\varepsilon(z)$ which satisfies the equation

$$\mathrm{i}\varepsilon\psi_\varepsilon'(z)=H(z)\psi_\varepsilon(z). \tag{2.21}$$

The eigenvalues $e_q^\sigma(t,\varepsilon)$, eigenvectors $\varphi_q^\sigma(t,\varepsilon)$ as well as the coefficients $c_q^\sigma(t,\varepsilon)$ in (2.13) have also analytic extensions on Ω, but these extensions are multivalued (see the proof of lemma 2.3). This fact is at the basis of the analysis of $\mathcal{P}(+,-)$ made by Landau and Lifshitz [18]. Let z_1 be the eigenvalue crossing of H which is the closest to the real axis in the ρ-distance. Let $t_1\leqslant t_2$ be two points of the real axis and γ a path in Ω going from t_1 to t_2 and such that the path composed of γ and then the portion of the real axis from t_2 to t_1 is a simple closed path encircling z_1 but no other eigenvalue crossing of H (see figure 1). At t_1 we have

$$\psi_\varepsilon(t_1)=\sum_{\sigma=\pm} c_q^\sigma(t_1,\varepsilon)\exp(-\mathrm{i}\varepsilon^{-1}\lambda_q^\sigma(t_1,\varepsilon))\varphi_q^\sigma(t_1,\varepsilon). \tag{2.22}$$

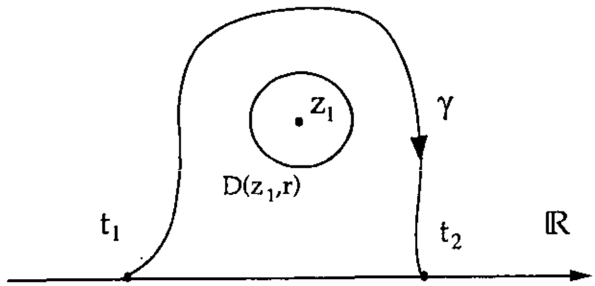

Figure 1. The path γ in Ω.

Each object on the right-hand side has an analytic extension along the path γ. Coming back to the real axis at t_2 we get

$$\psi_\varepsilon(t_2)=\sum_{\sigma=\pm}\tilde{c}_q^\sigma(t_2,\varepsilon)\exp(-\mathrm{i}\varepsilon^{-1}\tilde{\lambda}_q^\sigma(t_2,\varepsilon))\tilde{\varphi}_q^\sigma(t_2,\varepsilon) \tag{2.23}$$

where $\tilde{}$ means that (2.23) is the analytic extension of (2.22) along γ. This expression can be compared with

$$\psi_\varepsilon(t_2)=\sum_{\sigma=\pm}c_q^\sigma(t_2,\varepsilon)\exp(-\mathrm{i}\varepsilon^{-1}\lambda_q^\sigma(t_2,\varepsilon))\varphi_q^\sigma(t_2,\varepsilon) \tag{2.24}$$

defined earlier and which coincides with the analytic extension of (2.22) along the real axis from t_1 to t_2. By lemma 2.1, which is valid for H_q,

$$e_q^-(t_2,\varepsilon)=\tilde{e}_q^+(t_2,\varepsilon) \qquad e_q^+(t_2,\varepsilon)=\tilde{e}_q^-(t_2,\varepsilon) \tag{2.25}$$

since the eigenvalues of $H_q(z)$ are given by $\pm\frac{1}{2}\sqrt{\rho_q(z)}$, $\rho_q(z)=B_{q,1}^2(z,\varepsilon)+B_{q,2}^2(z,\varepsilon)+B_{q,3}^2(z,\varepsilon)$. The fact that φ_q^σ is an eigenvector of H_q is not affected by the analytic extension and therefore we can write

$$\tilde{\varphi}_q^+(t_2,\varepsilon)\equiv\exp(-\mathrm{i}\theta_q^{-,+}(\varepsilon))\varphi_q^-(t_2,\varepsilon) \tag{2.26}$$

and

$$\tilde{\varphi}_q^-(t_2,\varepsilon)\equiv\exp(-\mathrm{i}\theta_q^{+,-}(\varepsilon))\varphi_q^+(t_2,\varepsilon). \tag{2.27}$$

In general $\theta_q^{\sigma,\tau}$ is a complex number and depends on t_1, t_2 and γ. However, if we choose another path in Ω in the same homotopy class as γ, then (2.23) is not changed and, in particular, $\theta_q^{\sigma,\tau}$ in (2.26) and (2.27) is not modified. Since for any t_1 and t_2 on the real axis $\varphi_q^\sigma(t_2,\varepsilon)$ is the analytic continuation of $\varphi_q^\sigma(t_1,\varepsilon)$ along the real axis and since $\|\varphi_q^\sigma(t,\varepsilon)\|=1$ for all $t\in\mathbb{R}$, the *imaginary part* of $\theta_q^{\sigma,\tau}$ is *independent* of the choice of t_1 or t_2. Comparing (2.23) and (2.24) using (2.26) and (2.27) we get the fundamental relations

$$c_q^-(t_2,\varepsilon)=\exp(-\mathrm{i}\varepsilon^{-1}\tilde{\lambda}_q^+(t_2,\varepsilon)+\mathrm{i}\varepsilon^{-1}\lambda_q^-(t_2,\varepsilon))\exp(-\mathrm{i}\theta_q^{-,+})\tilde{c}_q^+(t_2,\varepsilon) \tag{2.28}$$

and

$$c_q^+(t_2,\varepsilon)=\exp(-\mathrm{i}\varepsilon^{-1}\tilde{\lambda}_q^-(t_2,\varepsilon)+\mathrm{i}\varepsilon^{-1}\lambda_q^+(t_2,\varepsilon))\exp(-\mathrm{i}\theta_q^{+,-})\tilde{c}_q^-(t_2,\varepsilon). \tag{2.29}$$

As for $\theta_q^{\sigma,\tau}$ the *imaginary part* of $\tilde{\lambda}_q^\sigma$ does not depend on the choice of t_1 and t_2 and remains unchanged if we choose another path in Ω in the same homotopy class as γ. This allows the following expressions to be given for these quantities. Let η be any simple closed path in Ω based at 0 (or any other point of the real axis) which encircles only the eigenvalue crossing z_1 of H and which is oriented clockwise. Then, for any $t_1\in\mathbb{R}$, $t_2\in\mathbb{R}$ and any path γ from t_1 to t_2 as above, we have

$$\operatorname{Im}\tilde{\lambda}_q^\sigma(t_2,\varepsilon)=\operatorname{Im}\int_\eta e_q^\sigma(z,\varepsilon)\,\mathrm{d}z \tag{2.30}$$

where $\int_\eta e_q^\sigma$ is the integral over η of the analytic continuation of e_q^σ along η. Similarly we can show that

$$\operatorname{Im}\theta_q^{+,-}=\operatorname{Im}\int_\eta\frac{B_{q,3}(B_{q,1}B'_{q,2}-B_{q,2}B'_{q,1})}{2\sqrt{\rho_q}(B_{q,1}^2+B_{q,2}^2)} \tag{2.31}$$

where here we must choose η so that $B_{q,1}^2(z,\varepsilon)+B_{q,2}^2(z,\varepsilon)\neq 0$ on η. To determine $\mathrm{Im}\,\theta_q^{-,+}$ we use the relation

$$\mathrm{Im}\,\theta_q^{+,-}=-\mathrm{Im}\,\theta_q^{-,+}. \tag{2.32}$$

From (2.29) we have

$$\begin{aligned}\mathscr{P}(+,-)&=|c_q^+(+\infty,\varepsilon)|^2\\&=\exp\left(2\varepsilon^{-1}\,\mathrm{Im}\int_\eta e_q^-(z,\varepsilon)\,\mathrm{d}z\right)\exp(2\,\mathrm{Im}\,\theta_q^{+,-}(\varepsilon))|\tilde{c}_q^-(+\infty,\varepsilon)|^2.\end{aligned} \tag{2.33}$$

We refer the reader to [13] for more details on this first part of the analysis and in particular for a proof of (2.31) and we come to the hard part of the analysis. It amounts to control the behaviour of $|\tilde{c}_q^-(+\infty,\varepsilon)|^2$ as a function of ε. This problem has been solved by Hwang and Pechukas [11]. Using their method we have:

Lemma 2.3. If conditions I–V hold and if $\psi_\varepsilon(z)$ satisfies the boundary condition (1.9) (or (2.19)), then

$$|\tilde{c}_q^-(+\infty,\varepsilon)|=1+\mathrm{O}(\varepsilon^q)$$

provided ε is small enough.

This lemma is proved in the next section. From it we get the main result of this paper.

Main result

$$\mathscr{P}(+,-)=\exp\left(2\varepsilon^{-1}\,\mathrm{Im}\int_\eta e_q^-(z,\varepsilon)\,\mathrm{d}z\right)\exp(2\,\mathrm{Im}\,\theta_q^{+,-}(\varepsilon))(1+\mathrm{O}(\varepsilon^q)). \tag{2.34}$$

From (2.34) it is easy to write the asymptotic expansion (1.12) by writing such an expansion for $e_q^+(z,\varepsilon)$ and $\theta_q^{+,-}(\varepsilon)$ using the explicit formula (2.31). This creates no difficulty since all expressions to be expanded are analytic in z and ε for $z\in\Omega$ and $|\varepsilon|$ small enough and uniformly bounded in z, $z\in\Omega$. Let us finish this section by giving the first two terms α_{-1} and α_0 of (1.12). We have

$$\alpha_{-1}=-2\,\mathrm{Im}\int_\eta e^-(z)\,\mathrm{d}z \tag{2.35}$$

where $\int_\eta e^-$ is the integral over η of the analytic continuation of the eigenvalue e^- of H along η. It is shown in [13] that

$$\mathrm{Im}\int_\eta e^-(z)\,\mathrm{d}z=-\mathrm{d}_\rho(z_1,\mathbb{R}) \tag{2.36}$$

where $\mathrm{d}_\rho(z_1,\mathbb{R})$ is the ρ-distance to the real axis of the closest eigenvalue crossing of H to the real axis in this distance. From perturbation theory we have that the first term in the expansion of $e_q^-(z,\varepsilon)$ in ε is $e^-(z)$ and that there is no term proportional to ε. Therefore α_0 is given by

$$\alpha_0=2\,\mathrm{Im}\int_\eta\frac{B_3(B_1B_2'-B_2B_1')}{2\sqrt{\rho}(B_1^2+B_2^2)}. \tag{2.37}$$

3. Proofs of lemmas

We prove by recurrence on q the following statements:

(i) for any integer $n \geqslant 0$ and any integer $q \geqslant 0$ there exist constants $D_{q,n}$ and $\varepsilon_*(q)$ (independent of n) such that for all $z \in \Omega$, all $\varepsilon \leqslant \varepsilon_*(q)$ and $k = 1, 2, 3$, $B_{q,k}(z, \varepsilon)$ are analytic on Ω,

$$|B_{q,k}(z, \varepsilon)| \leqslant D_{q,0}$$

and

$$(1 + |\mathrm{Re}\, z|)^{1+\alpha} |B_{q,k}^{(n)}(z, \varepsilon)| \leqslant D_{q,n}$$

where α is the constant appearing in condition II, $B_{0,k} \equiv B_k$, $k = 1, 2, 3$, and $B_{q,k}^{(n)} = \mathrm{d}^n/\mathrm{d}z^n\, B_{q,k}$;

(ii) there exists $\delta > 0$ such that

$$\inf_{\varepsilon \leqslant \varepsilon_*(q)} \inf_{z \in \Omega} |\rho_q(z, \varepsilon)| \geqslant \delta;$$

(iii) the variation of the argument of ρ_q around the boundary of any disc $D(z_k, r)$ (in the positive sense) is equal to 2π for any $\varepsilon \leqslant \varepsilon_*(q)$;

(iv) for any $q \geqslant 1$ and any $n \geqslant 0$ there exists a constant $F_{q,n}$ such that for $k = 1, 2, 3$

$$\sup_{\varepsilon \leqslant \varepsilon_*(q-1)} \sup_{z \in \Omega} (1 + |z|)^{1+\alpha} |B_{q,k}^{(n)}(z, \varepsilon) - B_{q-1,k}^{(n)}(z, \varepsilon)| \leqslant \varepsilon^q F_{q,n}$$

and for any $q \geqslant 1$ there exists a constant G_q such that

$$\sup_{\varepsilon \leqslant \varepsilon_*(q-1)} \sup_{z \in \Omega} |\rho_q(z, \varepsilon) - \rho_{q-1}(z, \varepsilon)| \leqslant \varepsilon^q G_q.$$

The validity of these four statements ensures the validity of lemmas 2.1 and 2.2.

Estimates (1.4) and (1.5) together with the remark following (1.5) imply that (i) is true for $q = 0$ with $\varepsilon_*(0) = \infty$. Clearly there exists $\delta > 0$ such that

$$\inf_{z \in \Omega} |\rho(z)| = \inf_{z \in \Omega} |\rho_0(z)| \geqslant 2\delta \tag{3.1}$$

so that (ii) is also verified. Since ρ_0 has exactly one zero and no pole inside any disc $D(z_k, r)$ (iii) follows, and finally (iv) is an immediate consequence of (i) and

$$\boldsymbol{B}_q = \boldsymbol{B} + \varepsilon \frac{\boldsymbol{B}_{q-1} \wedge \boldsymbol{B}'_{q-1}}{\rho_{q-1}} \tag{3.2}$$

when $q = 1$. Let us suppose that the four statements are true for $q = N - 1$ and let us prove them for $q = N$. It is immediate that (i) is true for $q = N$ and by (3.1), (3.2) and (i) the affirmation (ii) is also correct. Affirmation (iii) is a standard consequence of the argument principle. Indeed

$$|\rho_q(z, \varepsilon) - \rho(z)| \leqslant \varepsilon \times \text{constant} \tag{3.3}$$

uniformly in $z \in \Omega$ and $\varepsilon \leqslant \varepsilon_*(q-1)$. Let γ be the boundary of one disc (positively oriented). If ε is small enough we have for all $z \in \gamma$

$$\left| \frac{\rho_q(z, \varepsilon)}{\rho(z)} - 1 \right| < 1. \tag{3.4}$$

Let $G(z, \varepsilon) = \rho_q(z, \varepsilon)/\rho(z)$. This is a meromorphic function on some open set containing the disc and which has no zero and no pole on γ. The index of the image of γ by

G with respect to $z=0$ is zero since the image curve is contained in a disc of centre $z=1$ and radius smaller than 1. Thus

$$\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{G'(z,\varepsilon)}{G(z,\varepsilon)}\,\mathrm{d}z=\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{\rho_q'(z,\varepsilon)}{\rho_q(z,\varepsilon)}\,\mathrm{d}z-\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{\rho'(z)}{\rho(z)}\,\mathrm{d}z=0. \tag{3.5}$$

This proves (iii). Finally we have

$$\boldsymbol{B}_q=\boldsymbol{B}_{q-1}+\varepsilon\left(\frac{\boldsymbol{B}_{q-1}\wedge \boldsymbol{B}_{q-1}'}{\rho_{q-1}}-\frac{\boldsymbol{B}_{q-2}\wedge \boldsymbol{B}_{q-2}'}{\rho_{q-2}}\right) \tag{3.6}$$

and (iv) follows easily from the induction hypothesis.

It remains to prove lemma 2.3. Let $\Omega^-=\{z=t+\mathrm{i}s, |s|\leqslant a, t>t^-\}$ with t^- such that $\Omega^-\subset\Omega$. Let $\Omega^+=\{z=t+\mathrm{i}s, |s|\leqslant a, t>t^+\}$ with $\Omega^+\subset\Omega$ and $t^+>t^-$. On Ω^- we define $\Delta_q(z,\varepsilon)$ as the analytic continuation on Ω^- of the function

$$\int_0^t (e_q^-(t',\varepsilon)-e_q^+(t',\varepsilon))\,\mathrm{d}t'. \tag{3.7}$$

Let us recall that by our convention $e_0^\sigma(t,\varepsilon)=e^\sigma(t)$, $\sigma=+,-$. Condition V implies the existence of a path in S_a, $r\mapsto\gamma(r)$, parametrized by $r\in\mathbb{R}$, with the following properties:

(a) $\gamma(r)$ is contained in the upper half-plane:

$$\lim_{r\to\pm\infty}\operatorname{Im}\gamma(r)=s^\pm,\ s^\pm<a \qquad \lim_{r\to\pm\infty}\operatorname{Re}\gamma(r)=\pm\infty$$

(b) the open region between the real axis and γ contains the eigenvalue crossing z_1, but no other eigenvalue crossing of H;

(c) $r\mapsto\operatorname{Im}\Delta_0(\gamma(r))$ is a non-decreasing function of r, where Δ_0 is defined by analytic continuation along γ.

This is the main content of theorem 2.2 of [13]. Actually we can require that the path γ has the property:

(d) there exist r_1 and $r_2>r_1$ such that on $(-\infty, r_1]$ the function $\operatorname{Im}\Delta_0(\gamma(r))$ is constant, on $[r_1, r_2]$ its derivative with respect to r is strictly positive and on $[r_2,\infty)$ the function is again constant.

We can now fix the radius r of the small discs around the eigenvalue crossings of H. We choose r so small that the (Euclidian) distance from any eigenvalue crossing to γ is larger than $2r$, so that γ is entirely in Ω. The main step in the proof of lemma 2.3 is to show that there exists a path γ_q in Ω such that for all sufficiently small ε properties (a) and (b) are true and property (c) is true with Δ_q instead of Δ_0. Indeed, if we have such a path we proceed as follows. We make an analytic continuation of the coefficients $c_q^-(t,\varepsilon)$ and $c_q^+(t,\varepsilon)$ in Ω^-. We get functions $c_q^-(z,\varepsilon)$ and $c_q^+(z,\varepsilon)$ which are solutions of the differential equation

$$\begin{aligned} c_q^{+\prime}&=a_q^{+,+}c_q^+ + a_q^{+,-}\exp(-\mathrm{i}\varepsilon^{-1}\Delta_q)c_q^-\\ c_q^{-\prime}&=a_q^{-,+}\exp(\mathrm{i}\varepsilon^{-1}\Delta_q)c_q^+ + a_q^{-,-}c_q^- \end{aligned} \tag{3.8}$$

where in (3.8) $a_q^{\sigma,\tau}=a_q^{\sigma,\tau}(z,\varepsilon)$ is the analytic continuation of $a_q^{\sigma,\tau}(t,\varepsilon)$ in Ω^-. From (2.17) it is not completely obvious that $a_q^{\sigma,\tau}$ has an analytic continuation. However, we can see that this is the case by remarking that the function K_q has a single-valued analytic extension on Ω (see (2.8)). Let

$$U_q'(z,\varepsilon)=-\mathrm{i}K_q(z,\varepsilon)U_q(z,\varepsilon) \tag{3.9}$$

with

$$\lim_{t\to-\infty} U_q(t,\varepsilon)=\mathrm{I}. \tag{3.10}$$

This matrix is invertible on Ω^- since its determinant is one (K_q is traceless) and it is unitary on the real axis. From (2.11) we have

$$\varphi_q^\sigma(t,\varepsilon)=U_q(t,\varepsilon)\varphi^\sigma(-\infty) \qquad \sigma=+,- \tag{3.11}$$

so that we can write

$$a_q^{\sigma,\tau}(t,\varepsilon)=\mathrm{i}\langle (U_q(t,\varepsilon)^{-1})^*\varphi^\sigma(-\infty)|(K_q(t,\varepsilon)-K_{q-1}(t,\varepsilon))U_q(t,\varepsilon)\varphi^\tau(-\infty)\rangle. \tag{3.12}$$

This expression manifestly has an analytic continuation on Ω^-. Then we make an analytic continuation of c_q, Δ_q and $a_q^{\sigma,\tau}$ along γ_q and then in Ω^+ since by property (a) γ_q starts in Ω^- and ends in Ω^+. Of course (3.8) still holds. From lemma 2.2 and the definition of U_q we have in Ω^- or in Ω^+

$$\lim_{t\to\pm\infty} c_q^\sigma(t+\mathrm{i}s,\varepsilon)=\lim_{t\to\pm\infty} c_q^\sigma(t,\varepsilon)=c_q^\sigma(\pm\infty,\varepsilon) \qquad \sigma=+,-. \tag{3.13}$$

Therefore by property (b) the quantity $\tilde{c}_q^-(+\infty,\varepsilon)$ is in fact given by $c_q^-(+\infty,\varepsilon)$ as defined above. Thus it is sufficient to study $c_q^-(\gamma_q(r),\varepsilon)$ along γ_q for large values of r. Let $\dot\gamma_q(r)=\mathrm{d}/\mathrm{d}r\,\gamma_q(r)$. Along γ_q we have by writing $c_q^\sigma(r,\varepsilon)\equiv c_q^\sigma(\gamma_q(r),\varepsilon)$, $a_q^{\sigma,\tau}(r,\varepsilon)\equiv a_q^{\sigma,\tau}(\gamma_q(r),\varepsilon)$ and $\Delta_q(r,\varepsilon)\equiv\Delta_q(\gamma_q(r),\varepsilon)$

$$\begin{aligned} c_q^+(r,\varepsilon)=\int_{-\infty}^r \mathrm{d}r'\dot\gamma_q(r')a_q^{+,+}(r',\varepsilon)c_q^+(r',\varepsilon)+\int_{-\infty}^r \mathrm{d}r'\dot\gamma_q(r')a_q^{+,-}(r',\varepsilon)\\ \times\exp(-\mathrm{i}\varepsilon^{-1}\Delta_q(r',\varepsilon))c_q^-(r',\varepsilon)\end{aligned} \tag{3.14}$$

and

$$\begin{aligned} c_q^-(r,\varepsilon)=1+\int_{-\infty}^r \mathrm{d}r'\dot\gamma_q(r')a_q^{-,+}(r',\varepsilon)\exp(\mathrm{i}\varepsilon^{-1}\Delta_q(r',\varepsilon))c_q^+(r',\varepsilon)\\ +\int_{-\infty}^r \mathrm{d}r'\dot\gamma_q(r')a_q^{-,-}(r',\varepsilon)c_q^-(r',\varepsilon).\end{aligned} \tag{3.15}$$

Let $\boldsymbol{X}(r,\varepsilon)$ be the column vector whose components are

$$X_q^+(r,\varepsilon)=\exp(\mathrm{i}\varepsilon^{-1}\Delta_q(r,\varepsilon))c_q^+(r,\varepsilon) \qquad X_q^-(r,\varepsilon)=c_q^-(r,\varepsilon). \tag{3.16}$$

We can rewrite (3.14) and (3.15) as

$$\boldsymbol{X}_q(r,\varepsilon)=\begin{pmatrix}0\\1\end{pmatrix}+\int_{-\infty}^r \mathrm{d}r'\dot\gamma_q(r')A_q(r,r',\varepsilon)\boldsymbol{X}_q(r',\varepsilon) \tag{3.17}$$

where the matrix $A_q(r,r',\varepsilon)$ is

$$\begin{pmatrix} a_q^{+,+}(r',\varepsilon)\,\mathrm{e}^{\mathrm{i}\varepsilon^{-1}(\Delta_q(r,\varepsilon)-\Delta_q(r',\varepsilon))} & a_q^{+,-}(r',\varepsilon)\,\mathrm{e}^{\mathrm{i}\varepsilon^{-1}(\Delta_q(r,\varepsilon)-\Delta_q(r',\varepsilon))} \\ a_q^{-,+}(r',\varepsilon) & a_q^{-,-}(r',\varepsilon)\end{pmatrix}. \tag{3.18}$$

By lemma 2.2 and property (c) there exists a constant C such that for any $r\geqslant r'$

$$\int_{-\infty}^r \mathrm{d}r'|\dot\gamma_q(r')|\|A_q(r,r',\varepsilon)\|\leqslant\varepsilon^q C \tag{3.19}$$

and therefore

$$\lim_{r\to+\infty} c_q^-(r,\varepsilon)=1+\mathrm{O}(\varepsilon^q). \tag{3.20}$$

We now show that a path γ_q in Ω with properties (a), (b) and (c) always exists for all ε sufficiently small. Using property (d) of γ we choose for $r \in [r_1, r_2]$ $\gamma_q(r) = \gamma(r)$. This is possible if ε is small enough, since

$$\frac{\mathrm{d}}{\mathrm{d}r}\Delta_q(r, \varepsilon) = (e_q^-(\gamma(r), \varepsilon) - e_q^+(\gamma(r), \varepsilon))\dot{\gamma}_q(r). \tag{3.21}$$

Let $\zeta < r$ and

$$\Gamma_1(\zeta) = \left\{ z \in \Omega;\ \min_{r \in (-\infty, r_1]} |z - \gamma(r)| \leqslant \zeta \right\} \tag{3.22}$$

and let us consider the analytic continuation of Δ_0 on Γ_1. If ζ is small enough the function Δ_0 restricted on Γ_1 is injective. Let $\hat{\Gamma}_1$ be the image of Γ_1 by Δ_0. Since the image of the path $r \in (-\infty, r_1] \mapsto \gamma(r)$ by Δ_0 is horizontal, there exists $\eta > 0$ such that the strip $\hat{G}_1$,

$$\hat{G}_1 = \{z | \mathrm{Re}\, z \geqslant \mathrm{Re}\, \Delta_0(\gamma(r_1)), |\mathrm{Im}\, z - \mathrm{Im}\, \Delta_0(\gamma(r_1))| \leqslant \eta\} \tag{3.23}$$

is contained in $\hat{\Gamma}_1$. We denote by $\hat{g}_1^+$, $\hat{g}_1^-$ the two horizontal lines

$$\hat{g}_1^\pm = \{z | \mathrm{Im}\, z = \mathrm{Im}\, \Delta_0(\gamma(r_1)) \pm \eta, \mathrm{Re}\, z \geqslant \mathrm{Re}\, \Delta_0(\gamma(r_1))\} \tag{3.24}$$

and by G_1, g_1^+ and g_1^- the images of these sets by Δ_0^{-1}. We choose ε small enough such that for all $z \in G_1 \subset \Gamma_1$ we have

$$|\Delta_q(z, \varepsilon) - \Delta_0(z)| \leqslant \frac{\eta}{4}. \tag{3.25}$$

The path γ_q in G_1 is defined as the level line in G_1 of $\mathrm{Im}\, \Delta_q(z, \varepsilon)$, $\mathrm{Im}\, \Delta_q(z, \varepsilon) = \mathrm{Im}\, \Delta_q(\gamma(r_1), \varepsilon)$. Notice that

$$|\Delta_q(\gamma(r_1), \varepsilon) - \Delta_0(\gamma(r_1))| \leqslant \frac{\eta}{4} \tag{3.26}$$

and therefore by (3.25) it is impossible that γ_q intersects g_1^+ and g_1^-. We parametrize this level line by $r \in (-\infty, r_1]$. In a similar way we define $\gamma_q(r)$, $r \geqslant r_2$. This proves the existence of the path γ_q with properties (a), (b) and (c) and therefore the proof of lemma 2.3 is complete.

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